# Fluid Flow

Assume a basic conservation equation of the form

 (1)

where *c*, is the quantity of interest per unit volume, S is a source, D is diffusive flux and A is advective flux (symbols are also described in a separate file). The term on the left side of the equation, S, is a source term, and the first term on the right side is the divergence of the flux. The last term is the rate of change of the quantity stored per unit volume.

Fluid flow is simulated by assuming that mass and momentum are conserved. These assumptions allow two governing equations to be derived in terms of two unknown quantities, the mass flux or volumetric flux, and the pressure or hydraulic head.

### Preliminaries

Before getting started, it will be useful to define some of the basic terms, *displacement, strain, strain rate, and velocity* that are used to describe fluid flow.

#### Displacement

Displacement is a vector with units of [L] that describes the change in position of a point from one location to another



1 *u*d**i**x

2

*y,vd*

*x,ud*

*v*d**i**y

**u**

Displacement vector from location 1 to location 2

#### Velocity

Often displacement changes with time. The rate of displacement is the velocity, which is also a vector



and it has units of [L/T]. Velocity is



so



The *u, v, w* are commonly used for both displacement and velocity components, and **u** is used for the vector—Comsol does this, for example. I am using this convention but introducing the *d* subscript and **v** to distinguish displacement and velocity.

#### Strain

Strain is a measure of how much a material has deformed. There are two components, the normal and shear strains. Conceptually, the normal strain of a bar that is being pulled in tension is the ratio of the displacement of the bar to the original length. Shear strain is the change in angle between two lines. The units are dimensionless.

Strains can also be described in terms of gradients in displacement. A displacement gradient  is the rate at which the displacement in the x direction changes in the *y* direction, as shown below. This type of displacement measurement is a measure of the simple shear occurring in the plane x-z. The displacement gradient is the rate at which the displacement in the y direction changes in the x direction, and this is shown in the second panel below.

The shear strain is derived from the change in angle between three points during a deformation. Consider three points arranged so they are perpendicular. The points O, A, B below are an example. The angle between the lines connecting these points will be reduced if pt A is displaced to A’. The angle  is approximated by when is small. This is because is equal to tan() and when  is small. It follows that displacing A🡪A’ reduces the original angle by (Fig. 2b). The angle could also be changed by displacement of point B🡪B’. This changes the original angle by b, and . As a result, the total change in the original angle is the sum of the two displacement gradients, . The engineering shear strain is defined as half of the change in angle, which is the average of the two displacement gradients



Other shear strains can be defined similarly. The normal strain can be described in terms of displacement gradients, and it is simpler than the shear strain





*y,vd*

*x,ud*

*y,vd*

*x,ud*

*y,vd*

*x,u d*

Figure 1. Displacement gradients

*y,vd*

*x,ud*

*y,vd*

**

*x,ud*

*y,vd*

*x,ud*

A A’



B’



O B

A A’

O B

A

O B

Figure 2a Three reference points forming a right angle, A-O-B. Displacement of A🡪A’ reduces the angle by . Displacing B🡪B’ reduces the angle by 

Strain is a tensor that looks like this



#### Strain rate

The strain rate is a measure of how rapidly the material is deforming. It is the derivative of the strain with respect to time. So, for example, the normal strain rate is



However, the normal strain is itself a derivative of the displacements. So, writing the normal strain as a derivative and rearranging



Shows that the strain rate is equivalent to a gradient in velocity.

### Summary

**Displacement**: vector describing movement from one location to another [L]

**Velocity**: how rapidly material is moving; Displacement rate [L/T]

**Strain**: How much material has deformed; average of displacement gradients [--]

**Strain rate**: How fast material is deforming; velocity gradient [1/T]

## Conservation of Mass

We assume that the mass of the fluid is conserved within a control volume. , and it follows that , where *M* is taken as the mass in a control volume,. The fluid fills the control volume. The mass of fluid per unit volume of the control volume is *c* = , where  is the fluid density

Mass can be transported across the boundaries of the control volume when flow is occurring, but no mass flux occurs if no flow occurs. So, **D**=0 and the advective mass flux is

**A***=* **v***c* **=v** (2)

where **v** is the velocity vector vector. The velocity is equivalent to the volumetric flux. Checking the units of advective flux,  confirms that it is a mass flux. In this case, *L*c=*L*f because the fluid completely fills the control volume.

It is possible that mass could be added to the control volume as a source, say by injection or recharge, so 

S = M (3)

where M is the rate of mass produced by the source per unit volume. Substituting (2) and (3) into (1) gives

 (4)

This is the same as the conservation of mass during flow through a porous medium with the porosity and saturation set equal to unity.

The mass conservation equation written in Einstein notation is

 (4a)

There is only one repeated subscript, so writing out the equation in full gives

 (4b)

The special case where there are no sources and the system is at steady state is

 (4c)

## Conservation of Momentum

Conservation of momentum is expressed by setting , where *v* is velocity, and it follows that *c* is a momentum density. . The control volume consists of a continuous fluid.

Momentum flux is , which is the same units as stress.

Momentum can be advected, so applying (2) for a continuous fluid

 (5)

where **v** is the velocity of the fluid in the pore. The average velocity and the volumetric flux are equivalent.

The diffusive momentum flux creates a stress in the fluid. As an example, the shear stress yx is given by

 

Where  is the momentum diffusivity, or the kinematic viscosity. This simplifies where the density is uniform to



where  is the dynamic viscosity. Generalizing to other orientations gives the diffusive momentum flux as

 (6)

where (L2T2) is the stress on the fluid.

The rate of production of momentum internally is regarded to be a result of body forces . An example of a body force is **F** = g  where is the unit weight, which can be regarded as a force per volume or a pressure per unit length. The unit weight is the pressure exerted by a fluid per unit depth. The body force from the weight of the fluid is positive downward, so if the z coordinate is positive upward then the body force vector can be written as



where when in the downward direction, but it is zero in horizontal directions. Alternatively, the different components of the body force vector can be specified. In this case, the z component is -g, and the other components are zero. This latter approach is how effects resulting from the weight of a fluid are included in Comsol, for example.

Substituting into (1)

 (7)

Rearranging slightly

 (8)

The fluid density changes only slightly in most flows involving liquids. In these cases, the fluid is commonly assumed to be incompressible, so *d*/*dt* = 0, and we can divide through by  to get

 (9)

In Einstein convention, eq. 9 is



### Constitutive Equation for Newtonian Fluid

Fluids deform during flow and we need to include a way to describe this deformation in the analysis. The velocity of the fluid is

 (10)

where **i**x is a unit vector.

The stress in the fluid has components from the mean stress, or pressure, plus the shear stress or the viscous stresses



where the pressure is



Shear stress in a fluid depends on the gradient in velocity normal to the direction of the velocity. Consider flow parallel to a wall that lies in along the x-axis. The velocity is zero at the wall and the flow is essentially parallel to the wall. Then the shear stress is

 (11)

where  (nu) is the kinematic viscosity. Assuming the fluid density is uniform, or that whatever changes that do occur produce a gradient this is small enough to ignore, allows (11) to be revised to

 (12)

where  (mu) is the dynamic viscosity.

In a more general case where the velocity is not constrained by a nearby wall, shear stresses can be created by variations in velocity in the x direction. So,





*y,v*

*x,u*

*y,v*

*x,u*

*y,v*

*x,u*

Examples of velocity gradients that contribute to stress in a fluid.

This concept can be expanded to include normal stresses, so the stresses in a Newtonian fluid are



 (13)

where the pressure *P* is the mean stress



This can be written in a more compact way using index notation

 (14)

where the indices *i* and *j* are coordinate directions, *x,y,z*, and here *u* is the velocity in the direction of the index, x is the coordinate direction in the direction of the index. The term ij is kronicker delta, which is equal to 1 when *i = j*, but otherwise it is equal to zero.

The minus sign in front of the pressure is because the typical sign convention is for compressive stress to be negative and tensile stress to be positive.

The term is the divergence of the velocity. This is positive when the fluid is expanding, negative when it is contracting and when the fluid is incompressible and there are no sources or sinks. When  the velocity is divergence free.

The constitutive law simplifies for an incompressible fluid

 (15)

It is also useful to recognize that (13) contains the strain rate. Strain is defined in terms of displacement, and the displacement vector is



Notice that this notation is similar to the velocity vector, with the components of displacement defined using the same u, v, w that are used for the velocity vector and only the subscript distinguishes the two. Velocity is closely related to displacement,



and



Strain is the gradient of the displacement, where normal strains are



And shear strains are



This example shows how a normal strain rate is related to a gradient in velocity



It follows that

 (16a)

so the strain rate is a tensor

 (16b)

An alternative way of writing the strain rate tensor is

 (16c)

This can be visualized by recognizing  involves matrix multiplication, so



So



So (15) can be written as

 (17)

Substituting

 (18)

where the last term is the Laplacian of the velocity vector. Note that the Laplacian of a vector gives a tensor.

Substituting (38) into (34) give the Navier-Stokes equation for a linearly viscous fluid

 (19)

There are 5 main terms representing different processes.



The form of the Navier-Stokes equation given in (19) is convenient because it shows a compact form of the different terms. The equation can also be written using Einstein notation

 (20)

where the subscripts are used to indicate a direction, so the velocity components defined on page 1 are *u*1=*u*, *u*2=*v*, and *u*3=*w*. The repeated subscripts in (20) imply summation and it can be useful to write out the terms to clarify the Einstein notation







where it was assumed that the body force only acts in the *x*3 direction.

### Stokes Equation

The terms on the right side of eq (19) can be neglected when flow is very slow. This very slow flow is sometimes called “creeping” flow. Creeping flow is governed by the Stokes equation



Comsol uses the Stokes equation to solve for “creeping” flow.

### Non-dimensional form

The Navier-Stokes Equation can be written in dimensionless form, which can help with insights into some cases. The approach is to scale all the lengths to a characteristic length in a particular problem. This length could be the diameter of a conduit, or the length of a flowpath, but in any case we will label it, L. The velocities will be scaled to the average value for the problem, v’. The form of the Navier Stokes written above has units of Force/Volume, which has basic units of . So, if we divide both sides by

 (20)

Combining and defining dimensionless groups

 (21)

Recognizing the Reynolds number and dropping the primes gives the dimensionless form

 (22)

The Reynolds number scales the term that accounts for viscous dissipation. When Re is small, the importance of viscous dissipation is amplified. When Re is large and the flow is highly turbulent, however, the importance of viscous dissipation is diminished and the acceleration terms are relatively more important.

### Non-Newtonian Fluids

Water and oil are called Newtonian fluids because the shear stress in the fluids is proportional to the strain rate applied to the fluid. This is expressed in (17) where only the viscous stress components are used

 (23)

Newtonian fluids are also called “linearly viscous” because the shear stress is a linear function fo the strain rate and the dynamic viscosity is a constant of proportionality (Fig. ).

The rheological behavior of many common fluids differs from linearly viscous. The shear stress in these non-Newtonian fluids is a non-linear function of the strain rate. A simple and widely used approximation is a power law fluid where the shear stress is given by

 (24)

where *m* and *n* are parameters determined experimentally. Another way to view this behavior is to adopt (23) but assume that the viscosity  can change as a function of shear stress. In this case, the viscosity is interpreted as the slope of the curve relating shear stress to strain rate (Fig. ). The viscosity of a power-law fluid is

 (25)

using this approach.

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|  |
| Shear stress as function of shear strain rate for different types of fluids.  http://www.engineeringarchives.com/les\_fm\_newtoniannonnewtonian.html |

Fluids characterized by *n*<1 are called *pseudoplastic* or *shear-thinning* or *thixotropic*. This means that the faster the fluid is sheared, the lower the apparent viscosity. Fluids that behave this way include paint, ketchup, blood, magma, mud, and other slurries and gels. The shear-thinning aspect of these fluids is important to the behavior of these fluids. For example, the viscosity of paint is reduced when it is sheared with a paint brush, and this allows the fluid to spread as a thin layer. The apparent viscosity increases when the shear stops, however, and this causes the viscosity to increase, which prevents the paint from running down a vertical surface to form messy drips. Alternatively, the shear-thinning attribute makes it possible for these fluids to flow through conduits with less head drop than expected. The high shear along the wall of the conduit reduces the apparent viscosity there, which increases the flow rate compared to what would occur for a Newtonian fluid. This characteristic is important for the intrusion of magma into thin dikes, or the flow of blood through arteries.

Fluids characterized by n>1 are called dilatant or shear-thickening fluids. These fluids stiffen or dilate when sheared. This behavior is less common than thixotropic, but a well known example is corn starch and water. This mixture will flow at slow strain rates, but it is rigid a high strain rates. There are many videos on-line (http://www.ksl.com/?nid=148&sid=2694253) showing people running across pools of corn starch—the shear thickening rheology allows people to walk on water. Another shear-thickening fluid is a mixture of silica and ethylene glycol (http://dilatantfluids.weebly.com/2-structure.html). This fluid is combined with kevlar to improve the performance of body armor

The Carreau model is also used to describe the rheology of non-Newtonian fluids, particularly polymers. The apparent viscosity of a Carreau fluid is given by

 (26)

## Boundary Conditions

Inlet: Conditions where fluid enters; Outlet: condition where fluid leaves the system; Wall: condition along a boundary

### Inlet Conditions

#### Velocity

The velocity normal to the boundary is specified as U0



The minus sign is there because a positive flux is outward away from the boundary by default.

#### Pressure, no viscous stress

The pressure is set to a specified value, and the shear stresses normal to the boundary are set to zero. This would occur where the boundary is connected to a large body of static fluid held at constant pressure

*P*=*P*o



#### Laminar Inflow

The fluid velocity distribution at the boundary is assumed to be distributed as if there were a conduit upstream of the boundary that is long enough for the flow to become fully developed. For example, this would create a parabolic boundary condition for a straight-walled conduit normal to the boundary.

### Wall

#### No Slip

This condition requires that the velocity at the wall is zero. This is the typical assumption for the velocity of fluid at walls.



#### Slip

This condition specifies that the flow normal to the boundary is zero, but it allows flow parallel to the boundary. This is also called a “no penetration” condition because the fluid cannot penetrate the boundary. This condition could occur for inviscid flows, where the effect of viscosity can be ignored. This also could be used where the effect of the boundary is to prevent fluid from leaving the model domain, but the boundary is far away from the region of interest.



#### Sliding or Moving Wall

This condition allows the wall to slide or move. Sliding is a special case where the movement is only in the tangential direction.

#### Leaking Wall

This boundary condition represents a porous wall where the fluid leaks out but otherwise the boundary behaves like a solid surface.

# Turbulent Flow

The conceptual model for turbulent flow is that all of the variables describing the flow consist of a mean and a fluctuating value. Many applications are concerned with the mean value. For example, in many applications the mean value of the flow velocity is of primary importance. The fluctuations of variables is a result of turbulent vortices and they will also be important. Energy dissipated in the fluctuations of flow velocity are important because they are responsible for significant losses that are unrelated to useful work being done by the fluid. The fluctuations in velocity can be useful, however, because they provide an important mechanism for mixing components in chemical reactions, so turbulence will play an important role in some reaction rates.

### Mean and Fluctuating Terms

Based on this conceptual model, we will assume that the variables describing flow consist of a mean and a differential value. For example, the velocity vector is

 (27a)

where the bar indicates a mean value and the prime is a differential value. Similarly, density and pressure are

 (27b)



### Averaged Mass Conservation

The basic equations of conservation of mass (eq. 4) and momentum (9) are valid after we make the substitution in



We need another assumption. This one will be based on averaging. We will assume will assume that we can average properties over a time scale equivalent to the slowest turbulent fluctuations. Similarly, we will average over a length scale that allows us to ignore turbulent flucutations smaller than this scale. The resulting equations will be unable to resolve turbulent fluctuations explicitly, but the effects of these fluctuations will still be included in the averages.

The average conservation of mass equation is



Keep in mind that the bar symbol over an individual variable is the average value of that variable, and the bar over the sum is an average of the turbulent fluctuations.

In order to proceed, we need to recognize some properties of averages. The fluctuating quantities are fluctuating about zero, so their average is zero.





As a result, all of the terms involving the fluctuations go to zero, so the conservation of mass is

 (28)

This is the same form as the original conservation of mass.

### Averaged Momentum Conservation

The conservation of momentum is treated using the same approach, so starting with (20)

 (29)

And substituting (27) and averaging

 (29a)

The averaging process gives

 (29b)

Where all the primed terms drop out except the one to the right of the equal sign. This term needs some additional attention. The product rule gives

 (30)

The last term is the divergence of the velocity, which is

 (31)

So the last term drops out. Expanding the term on the left side of (30) gives



The terms containing are zero because averages to zero. However,  is not necessarily equal to zero. That is because if the two velocity components are not correlated, then the average of their product will not be zero. So

 (32)

Substituting (30) and (32) into (29) gives

 (33)

The first term on the right side is



because



So (33) becomes



And then grouping terms with the same derivative and dropping the averaging symbol

 (33)

This is essentially the same equation that we started with in (29) but in terms of averaged properties, but with one exception. The second term in the parentheses is new. This is the only term containing the fluctuating velocities.

### Reynold’s Stress and Closure

The first term in the parentheses gives the stresses in the fluid that result from the viscosity. The second term, the new one, is called the Reynold’s stress term because it resembles the regular stress term. This term was first derived by Osborne Reynold’s in the late 1800s. The Reynold’s stresses

 (34)

are a tensor, like the regular stresses, so there are 3 unique Reynold’s stresses in 2D and 6 Reynold’s stresses in 3D.

In an effort to simplify the problem of turbulent flow, we have developed a reasonable conceptual model that now leads to additional unknown variables. This will require additional equations before we can solve the problem. Having more unknowns than equations is called a “closure” problem. Accounting for the effects of turbulence using Reynold’s stresses causes a turbulence closure problem.

#### Eddy Viscosity

One way to start working on the closure problem is to make use of the similarity between the Reynold’s stress term and the viscous stresses, the two terms in parentheses in (33). Boussinesq (1892) proposed that the Reynold’s stresses could be related to the mean strain rates

 (35)

where t is the turbulence viscosity, or the eddy viscosity. This approach is conceptually appealing because it says that energy will be dissipated by turbulence much like it is dissipated by molecular motions characterized by viscosity. It is also appealing because it is essentially the same as the constitutive law that relates stresses to strain rates in a Newtonian fluid (eq. 13).

This offers a path forward because we can determine the strain rates, so we can solve for the Reynolds stresses using (35) if we can determine t.

### k- One Approach to Analyzing Turbulent Flow

There are many methods for estimating t. The *k*- (k-epsilon) method is useful for a variety of problems involving turbulent flow. It assumes that the turbulent viscosity is

 (35a)

which introduces, *k*, the turbulent kinetic energy, and , the turbulent dissipation rate. The actual turbulent kinetic energy is



where *m* is mass. Dividing through by *m* gives the turbulent kinetic energy per unit mass, which is

 (36)

Another important quantity is the turbulent energy dissipation rate

 (37)

This term is always positive and it is a measure of how rapidly the turbulent kinetic energy is being dissipated in the flow.

The k-e analysis also uses a slightly modified version of of (35), which includes k

 (37a)

The analysis outlined above introduced two new variables, the turbulent kinetic energy, *k*, and the turbulent dissipation rate, , in an effort to address the turbulence closure problem. Both the new variables are conserved, so we can derive governing equations using the approach that was used for other variables.

#### Turbulent kinetic energy, k

The dependent variable is the kinetic energy per unit volume,

We assume that the mass of the fluid is conserved within a control volume. , and it follows that , where *E* is the turbulent energy in a control volume,. The turbulent kinetic energy, k, has units of E/M, so the density term is needed.

The turbulent kinetic energy flux has both advection and diffusive terms. The advective energy flux is

**A***=* **v***c* **=v***k* (38)

where **v** is the velocity vector vector. The velocity is equivalent to the volumetric flux. Checking the units of advective flux,  confirms that it is an energy flux.

The diffusive energy flux

  (39)

Where the turbulent viscosity, t, appears as a coefficient of diffusion for kinetic energy. The turbulent viscosity is assumed to be



There are both sources and sinks for kinetic energy within the flow. One sink is described by the kinetic energy dissipation rate given above in (37). The source term is , where *G* is the turbulent energy production rate created by shear within the flow. This term is available here <http://staffweb.cms.gre.ac.uk/~ct02/research/thesis/node54.html>

Substituting gives a conservation equation for *k*

 (40)

#### Turbulent Dissipation Rate

The turbulent dissipation rate is also conserved and an approach similar to the one above gives *c*=and it follows that

 (41)

where *S* is the source term for the turbulent dissipation. The expressions (40) and (41) are refined in <http://www.cfd-online.com/Wiki/Standard_k-epsilon_model> and <http://en.wikipedia.org/wiki/K-epsilon_turbulence_model>

<http://www.mit.edu/~cuongng/Site/Publication_files/TurbulenceModeling_04NOV05.pdf>

and texts on fluid mechanics.

#### Implementation

The k-e analysis uses the four conservation equations, one for mass, momentum, turbulent kinetic energy and turbulent dissipation rate. The latter two are used to calculate the turbulent viscosity, which leads to closure of momentum conservation. This system of equations can be solved with appropriate boundary conditions to estimate pressure and flow velocities in some turbulent problems.