## Derivation of Navier Equation for Deformation of an Elastic Material

The equation relating stresses and deformations in an elastic material is based on conservation of momentum. This is the same equation that applies to fluids and it is obtained using the same approach that was used earlier

## Conservation of Momentum

Conservation of momentum is expressed by setting , where *v* is velocity, and it follows that *c* is an momentum density. . In a porous medium, , and in a continuous fluid we set *n*=1 to get

 (1)

Momentum flux is , which is the same units as stress

Momentum can be advected, so applying (3) for a continuous fluid

 (2)

The analogy to diffusion is

 (3)

where (L2T2) is the stress on the fluid.

The rate of production of momentum internally is regarded to be a result of body forces . An example of a body force is **F** = g  where is the unit weight, which can be regarded as a force per volume or a pressure per unit length. The unit weight is the pressure exerted by a fluid per unit depth.

Substituting into (7)

 (4)

Rearranging slightly

 (5)

The fluid density changes only slightly in solids and in many cases it can be assumed to be incompressible. Moreover, the velocities are typically small. If this is the case, then the inertial term involving v2 is negligible and can be ignored.

 (6)

### Constitutive Equation for Elastic Solid

Deformation in an elastic solid is described by the displacement vector

 (7)

where the d subscript is used to distinguish displacement from velocity, which is expressed as *u, v, w* above.

Stress and strain are related in 1D as

 (8)

And strain in the x direction is related to strain in the other directions through Poisson’s ratio, 

 (9)

This means that the strain in the x direction is the sum of strain components caused by loads in the x, y, and z directions. It follows that the strains in 3D are

 (10)

 (11)

 (12)

Solving eqs (4) simultaneously for stresses gives

 (13)

This can be reduced to a more compact form by using the Lame constants

 (14)

The term G is also called the shear modulus. A related constant, K, is the bulk modulus. 

Another important term is the volumetric strain, which is given by

 (15)

Substituting (6) and (7) into (5) gives Hooke’s Law as

 (16)

which can be written in a more compact form using index notation

 (17)

where *G* is the shear modulus and  is Lame’s constant.

Strains are defined as displacement gradients, so the normal strains are

 (18)

and shear strains are

 (19)

The volumetric strain is

 (20)

Recall that *u, v, w* is used to indicate the velocities in the *x, y, z* directions. The subscript *d* is used above to distinguish displacement from the velocity. These two quantities are closely related, however.

 (21)

Or

 (22)

Substituting (22) into the momentum equation (6) is

 (23)

where the term because the displacements are small. The term on the right side is used to analyze rapid changes, like seismic waves. For quasi-static problems, the term on the right side is zero, so

 (24)

It is convenient to express this equation in terms of displacements. To see how, expand (24)

 (25)

Another way to write this is

 (26)

The expression above uses a convention that says you increment and sum the repeated subscript in the derivative operator. In this case, the repeated subscript is *j*, so you would increment *j* through *x, y, z* while holding *i* constant, and then sum the three resulting terms. This would give you one of the previous equations. Changing *i* and repeating the process will give the other two equations.

Using (17) and substituting into (25)



Grouping terms and expanding each equation in (25)







Substituting in the definition of strain in terms of displacements in (19) and (20)

simplifying the right side in several steps like this









Substituting back into (25) and repeating for the other two equations gives

 (27)





The three equations above can be written using subscript notation as

 (28)

where each equation is represented by a different value of i, and the repeated subscript j in the derivative means to increment through *x, y, z* and sum the three resulting terms. This is the Navier Equation.