**Single-phase flow through porous media**

Assume a basic conservation equation of the form

 (1)

where *c*, is the quantity of interest per unit volume, S is a source, D is diffusive flux and A is advective flux (symbols are also described in a separate file). The term on the left side of the equation, S, is a source term, and the first term on the right side is the divergence of the flux. The last term is the rate of change of the quantity stored per unit volume.

Flow through porous media is simulated by assuming that mass and momentum are conserved. These assumptions allow two governing equations to be derived in terms of two unknown quantities, the mass or volumetric flux and the pressure or hydraulic head.

**Conservation of Mass**

This problem consists of flow of a fluid in a porous medium, so we assume that the mass of the fluid is conserved within a control volume that contains the porous medium. , and it follows that , where *M* is taken as the mass in a control volume,. The fluid shares the control volume with solid grains and perhaps with another fluid. The mass of fluid per unit volume of the control volume is *c* = *S*e, where f is the fluid density,  is porosity of the porous medium,, and *S*e is the degree of saturation of the fluid . *c* = *S*e 

Mass can be transported across the boundaries of the control volume when flow is occurring, but no mass flux occurs if no flow occurs. So, **D**=0 and the advective mass flux is

**A***=* **q***c***/**(*S*e) **=q** (2)

where **q** is the volumetric flux vector, *S*e is the degree of saturation of phase of interest, and  is porosity. The terms  in the denominator appear because of the porous media and they will be absent (equal to 1) when a continuous fluid is considered. For reference, the volumetric water content is , and . Checking the units of advective flux,  confirms that it is a mass flux.

It is possible that mass could be added to the control volume as a source, say by injection or recharge, so 

S = M (3)

where M is the rate of mass produced by the source per unit volume. Substituting (2) and (3) into (1) gives

 (4)

**Conservation of Momentum**

Conservation of momentum is expressed by setting , where *v* is velocity, and it follows that *c* is an momentum density. . The control volume will initially consist of a single pore, and then we will average that result over the REV.

Momentum flux is , which is the same units as stress.

Momentum can be advected, so applying (2) for a continuous fluid

 (5)

where **v** is the velocity of the fluid in the pore.

The diffusive momentum flux creates a stress in the fluid. As an example, the shear stress yx is given by

 

Where  is the momentum diffusivity, or the kinematic viscosity. This simplifies where the density is uniform to



where  is the dynamic viscosity. Generalizing to other orientations gives the diffusive momentum flux as

 (6)

where (L2T2) is the stress on the fluid.

The rate of production of momentum internally is assumed to be a result of body forces . An example of a body force is **F** = g  where is the unit weight, which can be regarded as a force per volume or a pressure per unit length. The unit weight is the pressure exerted by a fluid per unit depth.

Substituting into (7)

 (7)

Rearranging slightly

 (8)

The fluid density changes only slightly in most flows involving liquids. In these cases, the fluid is commonly assumed to be incompressible, so *d*/*dt* = 0, and we can divide through by  to get

 (9)

**Constitutive Equation for Newtonian Fluid**

Fluids deform during flow and we need to include a way to describe this deformation in the analysis. The velocity of the fluid is

 (10)

where **i**x is a unit vector.

Shear stress in a linearly viscous fluid is proportional to the gradient in velocity perpendicular to the flow, so

 (11)

where  (nu) is the kinematic viscosity. Assuming the fluid density is uniform, or that whatever changes that do occur produce a gradient this is small enough to ignore, allows (11) to be revised to

 (12)

where  (mu) is the dynamic viscosity.

This concept can be expanded to include normal stresses due to pressure, so the stresses in a Newtonian fluid are

 (13)

where *P*’ is the potential, *P*’= *P* + *gz*. This can be written in a more compact way using index notation

 (14)

where the indices *i* and *j* are coordinate directions, *x,y,z*, and here *u* is the velocity in the direction of the index, x is the coordinate direction in the direction of the index. The term ij is kronicker delta, which is equal to 1 when *i = j*, but otherwise it is equal to zero.

The minus sign in front of the pressure is because the typical sign convention is for compressive stress to be negative and tensile stress to be positive.

 (15)

Where the is shear stress in the fluid. Shear stress is indicated above as , but will be used instead in what follows for convenience.

Substituting (38) into (34) give the Navier-Stokes equation for a linearly viscous fluid

 (16)

There are 5 main terms representing different processes.



**Simplification for Porous Media**

Navier-Stokes equation is versatile, but it can be cumbersome to solve and the analysis of flow through porous media commonly uses a simpler form of the momentum balance that is easier to solve.

Flow through porous media is generally slow, so the velocity is low and it changes slowly. This means that the acceleration of the fluid (the time derivative) is small and we will ignore it here. Body forces are also ignored, leaving

 (17)

A consequence of small velocities is that the square of the velocity is even smaller, so the right side of (16) is assumed negligible in porous media. This leaves

 (18)

This equation must apply as an average over a representative volume of characteristic length, *L*, so

 (18a)

where the pointy brackets mean that the quantity is averaged over the control volume. The pressure gradient is uniform over the characteristic length, so  and

 (18b)

The average of the divergence of the shear stress in the fluid over a representative volume is determined using

 (19)

The volume average of the divergence of  can be written in terms of areal integral using Gauss’s theorem from calculus

 (20)

where *A*pore is the area of a representative pore. is the shear stress normal to the boundary, and we have assumed that shear stress on the external boundary of the control volume can be ignored. The volume average for a cylindrical pore is approximated by assuming the shear stress is roughly uniform

 (21)

where *d* is the pore diameter, *L* is length of the REV. The pore is assumed to be cylindrical in the intermediate step above, but the final result is valid for pores with irregular shapes. It follows that

 (22)

where  on the right hand side is the shear stress on the wall of the pore. Generalizing from the Hagen-Poiseuille equation, the pressure gradient during laminar flow in a pore is

 (23)

where *q* is volumetric flux,  is dynamic viscosity, *G* is a geometric factor accounting for the pore geometry.

|  |
| --- |
| *d*x |
| Fig. 1. Force balance on a differential fluid element with a circular cross-section |

A force balance on an incremental slice of fluid in a circular pore includes the fluid pressure and shear stresses on the walls (Fig 1). Balancing forces



which simplifies to



so

 (24)

Equating to (23)

 (25)

And solving for the shear stress

 (26)

From (22) the average shear stress is

 (27)

Rearranging (18a)

 (28)

And then substituting (27)

 (29)

rearranging

 (30)

simplifying

 (31)

where *k* is permeability. Eq. 31 is valid for horizontal flow, but we need to include the effect of gravity for vertical flows. To do this, we return to (16) and include the body force in the fluid

**F**=-g (31a)

A body force is a force per unit volume. Eq (31) is a compact way of writing the body force vector **F** = [Fx, Fy, Fz]=[0, 0, -*g*]*.*  This is because **= 1 when **F** is upward parallel to the z axis, but it is zero when **F** is in directions other than upward. When the body force is included, (16) is simplified to



and then



The analysis then proceeds as above, but is replaced with . The result is that (31) becomes

 (32)

which is Darcy’s Law. In some applications it is convenient to use Darcy’s Law written in terms of hydraulic head,

 (33)

 (34a)

where the hydraulic conductivity is

 (34b)

This result shows that the conservation of momentum for a porous medium is given by Darcy’s Law.

The result above (41) applies to cases where fluid density could vary spatially. Assuming the fluid density is uniform gives

 (35)

which assumes the permeability is isotropic. When the formation is anisotropic, then *k* is a tensor of the form



in 3D. This tensor is symmetric, so *k*21=*k*12. This means that there are 6 independent terms in the tensor, instead of 9.

This has important implications for Darcy’s Law. It means that for anisotropic conditions flow in one direction can be driven by a gradient in a normal direction. Darcy’s Law becomes



In matrix notation

 (36a)

In Einstein notation

 (36b)

or using the del operator

 (36c)

**Modification of Mass Conservation, Storage change in Porous Media**

The sections above lead to two expressions for conservation of mass and momentum. The conservation of mass is given by

 (45)

This equation can be simplified in a way that facilitates a solution by expanding the second term on the right side using the chain rule



And applying the chain rule again

 (46)

The fluid compressibility is defined as

 (47)

And assuming the compressibility of the aquifer skeleton results from a change in porosity leads to the compressibility of the aquifer as

 (48)

where the fluid bulk modulus is *K*f and the porous media bulk modulus is *K*p.

The specific moisture capacity is the change in volumetric moisture content,  per unit change in head under partially saturated conditions

 (49)

The volumetric water content is related to the degree of saturation as  = *Se*. Expanding using the chain rule

 (50)

because  and . Substituting (46 - 49) into (45)



and grouping

 (51)

The specific storage of the aquifer defined in terms of pressure is

 (52)

so

 (53)

And from the chain rule we get the rate of change of storage

 (54)

Now we take this result and substitute (54) into (45) to give the mass conservation expression

 (55)

Expressions (55) and (44) are used in Comsol. *C*m=0, and *S*e=1 for saturated conditions.

**Representation as a Diffusion Equation**

The two governing equations derived from conservation of momentum and mass can be combined by substituting (44) into (55)

 (56a)

In terms of hydraulic head

 (56b)

And for saturated conditions where the fluid is uniform density

 (56c)

where is the specific storage in terms of pressure head.

One convenient aspect of (56) is that there is only one dependent variable, either *P* or *h*, whereas in (55) there are two dependent variables. You would need another equation to solve (55) because it has two dependent variables. If you use (55) then you would also need to use Darcy’s Law, or an equivalent, to obtain a solution. Darcy’s Law is already included in (56), which is why it is commonly used.

**Parameters**

The analyses outlined above include parameters describing properties of fluids and porous media.

**Dynamic viscosity**. ****  Depends on temperature and pressure.

1 Poise = 0.1 Pa s

at standard PT conditions,

water: 0.001 Pa s = 0.01 Poise = 1 centipoise.

air: 1.8x10-5 Pa s = roughly 1/50 of water

**Kinematic viscosity. ** =

1 Stoke =10-4m2/s

water: 10-6 m2/ s = 1centiStoke.

air: ~10x10-6 m2/ s

**Density, **

at standard P and T

water: 1000 kg/m3

air: 1.2 kg/m3

**Permeability, *k***

10-12 m2 = 1m2 = 1Darcy

See chart below for ranges. Some typical values:

*k*sand: 10-12 m2 = 1m2 = 1Darcy

*k*silt: 10-14 m2 = 10mD

*k*clay 10-16 m2 = 100D

kwood: ~10-14m2

**Porosity, **

Well sorted sand: 0.25-0.5

Poorly sorted sand: 0.15-0.3

Clay: 0.4-0.6

Crystalline rock: 0.001-0.01

Fractured rock: 0.01-0.05

**Compressibility, **



| **Viscosity of**[**liquids**](http://en.wikipedia.org/wiki/Liquid) **(at 25 °**[**C**](http://en.wikipedia.org/wiki/Celsius)**unless otherwise specified) wikipedia** | | |
| --- | --- | --- |
| **Liquid :** | **Viscosity**  **[Pa·s]** | **Viscosity**  **[cP=mPa·s]** |
| [water](http://en.wikipedia.org/wiki/Water) | 8.94×10−4 | 0.894 |
| [sulfuric acid](http://en.wikipedia.org/wiki/Sulfuric_acid)[[24]](http://en.wikipedia.org/wiki/Viscosity#cite_note-CRC-24) | 2.42×10−2 | 24.2 |
| [propanol](http://en.wikipedia.org/wiki/Propan-1-ol)[[24]](http://en.wikipedia.org/wiki/Viscosity#cite_note-CRC-24) | 1.945×10−3 | 1.945 |
| [pitch](http://en.wikipedia.org/wiki/Pitch_(resin)) | 2.3×108 | 2.3×1011 |
| [olive oil](http://en.wikipedia.org/wiki/Olive_oil) | .081 | 81 |
| [nitrobenzene](http://en.wikipedia.org/wiki/Nitrobenzene)[[24]](http://en.wikipedia.org/wiki/Viscosity#cite_note-CRC-24) | 1.863×10−3 | 1.863 |
| [motor oil](http://en.wikipedia.org/wiki/Motor_oil) SAE 40 (20 °C)[[13]](http://en.wikipedia.org/wiki/Viscosity#cite_note-physicsinfo-13) | 0.319 | 319 |
| [motor oil](http://en.wikipedia.org/wiki/Motor_oil) SAE 10 (20 °C)[[13]](http://en.wikipedia.org/wiki/Viscosity#cite_note-physicsinfo-13) | 0.065 | 65 |
| [methanol](http://en.wikipedia.org/wiki/Methanol)[[24]](http://en.wikipedia.org/wiki/Viscosity#cite_note-CRC-24) | 5.44×10−4 | 0.544 |
| [mercury](http://en.wikipedia.org/wiki/Mercury_(element))[[24]](http://en.wikipedia.org/wiki/Viscosity#cite_note-CRC-24) | 1.526×10−3 | 1.526 |
| [liquid nitrogen](http://en.wikipedia.org/wiki/Liquid_nitrogen) @ 77K | 1.58×10−4 | 0.158 |
| [HFO-380](http://en.wikipedia.org/wiki/Fuel_oil) | 2.022 | 2022 |
| [glycerol](http://en.wikipedia.org/wiki/Glycerol) (at 20 °[C](http://en.wikipedia.org/wiki/Celsius))[[25]](http://en.wikipedia.org/wiki/Viscosity#cite_note-25) | 1.2 | 1200 |
| [ethylene glycol](http://en.wikipedia.org/wiki/Ethylene_glycol) | 1.61×10−2 | 16.1 |
| [ethanol](http://en.wikipedia.org/wiki/Ethanol)[[24]](http://en.wikipedia.org/wiki/Viscosity#cite_note-CRC-24) | 1.074×10−3 | 1.074 |
| [corn syrup](http://en.wikipedia.org/wiki/Corn_syrup)[[24]](http://en.wikipedia.org/wiki/Viscosity#cite_note-CRC-24) | 1.3806 | 1380.6 |
| [castor oil](http://en.wikipedia.org/wiki/Castor_bean)[[24]](http://en.wikipedia.org/wiki/Viscosity#cite_note-CRC-24) | 0.985 | 985 |
| [benzene](http://en.wikipedia.org/wiki/Benzene)[[24]](http://en.wikipedia.org/wiki/Viscosity#cite_note-CRC-24) | 6.04×10−4 | 0.604 |
| [acetone](http://en.wikipedia.org/wiki/Acetone)[[24]](http://en.wikipedia.org/wiki/Viscosity#cite_note-CRC-24) | 3.06×10−4 | 0.306 |

Permeability and Hydraulic Conductivity

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| http://faculty.gg.uwyo.edu/neil/teaching/geohydro/lect_images/FIG6_5.JPG  microDarcy    milliDarcy    Darcy |
| http://faculty.gg.uwyo.edu/neil/teaching/geohydro/lect\_images/FIG6\_5.JPG |

|  |  |
| --- | --- |
| **Vertical, drained compressibilities**[[2]](http://en.wikipedia.org/wiki/Compressibility#cite_note-2) | |
| **Material** | ***β* (m²/N or Pa-1)** |
| Plastic clay | 2×10–6 – 2.6×10–7 |
| Stiff clay | 2.6×10–7 – 1.3×10–7 |
| Medium-hard clay | 1.3×10–7 – 6.9×10–8 |
| Loose sand | 1×10–7 – 5.2×10–8 |
| Dense sand | 2×10–8 – 1.3×10–8 |
| Dense, sandy gravel | 1×10–8 – 5.2×10–9 |
| Rock, fissured | 6.9×10–10 – 3.3×10–10 |
| Rock, sound | <3.3×10–10 |
| Water at 25 °C (undrained)[[3]](http://en.wikipedia.org/wiki/Compressibility#cite_note-3) | 4.6×10–10 |