# Fluid Flow

Assume a basic conservation equation of the form

 (1)

where *c*, is the quantity of interest per unit volume, S is a source, D is diffusive flux and A is advective flux (symbols are also described in a separate file). The term on the left side of the equation, S, is a source term, and the first term on the right side is the divergence of the flux. The last term is the rate of change of the quantity stored per unit volume.

Fluid flow is simulated by assuming that mass and momentum are conserved. These assumptions allow two governing equations to be derived in terms of two unknown quantities, the mass flux or volumetric flux, and the pressure or hydraulic head.

### Preliminaries

Before getting started, it will be useful to define some of the basic terms, *displacement, strain, strain rate, and velocity* that are used to describe fluid flow.

#### Displacement

Displacement is a vector with units of [L] that describes the change in position of a point from one location to another



1 *u*d**i**x

2

*y,vd*

*x,ud*

*v*d**i**y

**u**

Displacement vector from location 1 to location 2

#### Velocity

Often displacement changes with time. The rate of displacement is the velocity, which is also a vector



and it has units of [L/T]. Velocity is



so



The *u, v, w* are commonly used for both displacement and velocity components, and **u** is used for the vector—Comsol does this, for example. I am using this convention but introducing the *d* subscript and **v** to distinguish displacement and velocity.

#### Strain

Strain is a measure of how much a material has deformed. There are two components, the normal and shear strains. Conceptually, the normal strain of a bar that is being pulled in tension is the ratio of the displacement of the bar to the original length. Shear strain is the change in angle between two lines. The units are dimensionless.

Strains can also be described in terms of gradients in displacement. A displacement gradient  is the rate at which the displacement in the x direction changes in the *y* direction, as shown below. This type of displacement measurement is a measure of the simple shear occurring in the plane x-z. The displacement gradient is the rate at which the displacement in the y direction changes in the x direction, and this is shown in the second panel below.

The shear strain is derived from the change in angle between three points during a deformation. Consider three points arranged so they are perpendicular. The points O, A, B below are an example. The angle between the lines connecting these points will be reduced if pt A is displaced to A’. The angle  is approximated by when is small. This is because is equal to tan() and when  is small. It follows that displacing A🡪A’ reduces the original angle by (Fig. 2b). The angle could also be changed by displacement of point B🡪B’. This changes the original angle by b, and . As a result, the total change in the original angle is the sum of the two displacement gradients, . The engineering shear strain is defined as half of the change in angle, which is the average of the two displacement gradients



Other shear strains can be defined similarly. The normal strain can be described in terms of displacement gradients, and it is simpler than the shear strain





*y,vd*

*x,ud*

*y,vd*

*x,ud*

*y,vd*

*x,u d*

Figure 1. Displacement gradients

*y,vd*

*x,ud*

*y,vd*

**

*x,ud*

*y,vd*

*x,ud*

A A’



B’



O B

A A’

O B

A

O B

Figure 2a Three reference points forming a right angle, A-O-B. Displacement of A🡪A’ reduces the angle by . Displacing B🡪B’ reduces the angle by 

Strain is a tensor that looks like this



#### Strain rate

The strain rate is a measure of how rapidly the material is deforming. It is the derivative of the strain with respect to time. So, for example, the normal strain rate is



However, the normal strain is itself a derivative of the displacements. So, writing the normal strain as a derivative and rearranging



Shows that the strain rate is equivalent to a gradient in velocity.

### Summary

**Displacement**: vector describing movement from one location to another [L]

**Velocity**: how rapidly material is moving; Displacement rate [L/T]

**Strain**: How much material has deformed; average of displacement gradients [--]

**Strain rate**: How fast material is deforming; velocity gradient [1/T]

## Conservation of Mass

We assume that the mass of the fluid is conserved within a control volume. , and it follows that , where *M* is taken as the mass in a control volume,. The fluid fills the control volume. The mass of fluid per unit volume of the control volume is *c* = , where  is the fluid density

Mass can be transported across the boundaries of the control volume when flow is occurring, but no mass flux occurs if no flow occurs. So, **D**=0 and the advective mass flux is

**A***=* **q***c* **=q** (2)

where **q** is the volumetric flux vector. Checking the units of advective flux,  confirms that it is a mass flux.

It is possible that mass could be added to the control volume as a source, say by injection or recharge, so 

S = M (3)

where M is the rate of mass produced by the source per unit volume. Substituting (2) and (3) into (1) gives

 (4)

This is the same as the conservation of mass during flow through a porous medium with the porosity and saturation set equal to unity.

## Conservation of Momentum

Conservation of momentum is expressed by setting , where *v* is velocity, and it follows that *c* is a momentum density. . The control volume consists of a continuous fluid.

Momentum flux is , which is the same units as stress.

Momentum can be advected, so applying (2) for a continuous fluid

 (5)

where **v** is the velocity of the fluid in the pore. The average velocity and the volumetric flux are equivalent.

The diffusive momentum flux creates a stress in the fluid. As an example, the shear stress syx is given by

 

Where  is the momentum diffusivity, or the kinematic viscosity. This simplifies where the density is uniform to



where  is the dynamic viscosity. Generalizing to other orientations gives the diffusive momentum flux as

 (6)

where (L2T2) is the stress on the fluid.

The rate of production of momentum internally is regarded to be a result of body forces . An example of a body force is **F** = g  where is the unit weight, which can be regarded as a force per volume or a pressure per unit length. The unit weight is the pressure exerted by a fluid per unit depth. The body force from the weight of the fluid is positive downward, so if the z coordinate is positive upward then the body force vector can be written as



where when in the downward direction, but it is zero in horizontal directions. Alternatively, the different components of the body force vector can be specified. In this case, the z component is -g, and the other components are zero. This latter approach is how effects resulting from the weight of a fluid are included in Comsol, for example.

Substituting into (1)

 (7)

Rearranging slightly

 (8)

The fluid density changes only slightly in most flows involving liquids. In these cases, the fluid is commonly assumed to be incompressible, so *d*/*dt* = 0, and we can divide through by  to get

 (9)

### Constitutive Equation for Newtonian Fluid

Fluids deform during flow and we need to include a way to describe this deformation in the analysis. The velocity of the fluid is

 (10)

where **i**x is a unit vector.

The stress in the fluid has components from the mean stress, or pressure, plus the shear stress or the viscous stresses



where the pressure is



Shear stress in a fluid depends on the gradient in velocity normal to the direction of the velocity. Consider flow parallel to a wall that lies in along the x-axis. The velocity is zero at the wall and the flow is essentially parallel to the wall. Then the shear stress is

 (11)

where  (nu) is the kinematic viscosity. Assuming the fluid density is uniform, or that whatever changes that do occur produce a gradient this is small enough to ignore, allows (11) to be revised to

 (12)

where  (mu) is the dynamic viscosity.

In a more general case where the velocity is not constrained by a nearby wall, shear stresses can be created by variations in velocity in the x direction. So,





*y,v*

*x,u*

*y,v*

*x,u*

*y,v*

*x,u*

Examples of velocity gradients that contribute to stress in a fluid.

This concept can be expanded to include normal stresses, so the stresses in a Newtonian fluid are



 (13)

where the pressure *P* is the mean stress



This can be written in a more compact way using index notation

 (14)

where the indices *i* and *j* are coordinate directions, *x,y,z*, and here *u* is the velocity in the direction of the index, x is the coordinate direction in the direction of the index. The term ij is kronicker delta, which is equal to 1 when *i = j*, but otherwise it is equal to zero.

The minus sign in front of the pressure is because the typical sign convention is for compressive stress to be negative and tensile stress to be positive.

The term is the divergence of the velocity. This is positive when the fluid is expanding, negative when it is contracting and when the fluid is incompressible and there are no sources or sinks. When  the velocity is divergence free.

The constitutive law simplifies for an incompressible fluid

 (15)

It is also useful to recognize that (13) contains the strain rate. Strain is defined in terms of displacement, and the displacement vector is



Notice that this notation is similar to the velocity vector, with the components of displacement defined using the same u, v, w that are used for the velocity vector and only the subscript distinguishes the two. Velocity is closely related to displacement,



and



Strain is the gradient of the displacement, where normal strains are



And shear strains are



This example shows how a normal strain rate is related to a gradient in velocity



It follows that

 (16a)

so the strain rate is a tensor

 (16b)

An alternative way of writing the strain rate tensor is

 (16c)

This can be visualized by recognizing  involves matrix multiplication, so



So



So (15) can be written as

 (17)

Substituting

 (18)

where the last term is the Laplacian of the velocity vector. Note that the Laplacian of a vector gives a tensor.

Substituting (38) into (34) give the Navier-Stokes equation for a linearly viscous fluid

 (19)

There are 5 main terms representing different processes.



### Stokes Equation

The terms on the right side of eq (19) can be neglected when flow is very slow. This very slow flow is sometimes called “creeping” flow. Creeping flow is governed by the Stokes equation



Comsol uses the Stokes equation to solve for “creeping” flow.

### Non-dimensional form

The Navier-Stokes Equation can be written in dimensionless form, which can help with insights into some cases. The approach is to scale all the lengths to a characteristic length in a particular problem. This length could be the diameter of a conduit, or the length of a flowpath, but in any case we will label it, L. The velocities will be scaled to the average value for the problem, v’. The form of the Navier Stokes written above has units of Force/Volume, which has basic units of . So, if we divide both sides by

 (20)

Combining and defining dimensionless groups

 (21)

Recognizing the Reynolds number and dropping the primes gives the dimensionless form

 (22)

The Reynolds number scales the term that accounts for viscous dissipation. When Re is small, the importance of viscous dissipation is amplified. When Re is large and the flow is highly turbulent, however, the importance of viscous dissipation is diminished and the acceleration terms are relatively more important.

### Non-Newtonian Fluids

Water and oil are called Newtonian fluids because the shear stress in the fluids is proportional to the strain rate applied to the fluid. This is expressed in (17) where only the viscous stress components are used

 (23)

Newtonian fluids are also called “linearly viscous” because the shear stress is a linear function fo the strain rate and the dynamic viscosity is a constant of proportionality (Fig. ).

The rheological behavior of many common fluids differs from linearly viscous. The shear stress in these non-Newtonian fluids is a non-linear function of the strain rate. A simple and widely used approximation is a power law fluid where the shear stress is given by

 (24)

where *m* and *n* are parameters determined experimentally. Another way to view this behavior is to adopt (23) but assume that the viscosity  can change as a function of shear stress. In this case, the viscosity is interpreted as the slope of the curve relating shear stress to strain rate (Fig. ). The viscosity of a power-law fluid is

 (25)

using this approach.

|  |
| --- |
|  |
| Shear stress as function of shear strain rate for different types of fluids.  http://www.engineeringarchives.com/les\_fm\_newtoniannonnewtonian.html |

Fluids characterized by *n*<1 are called *pseudoplastic* or *shear-thinning* or *thixotropic*. This means that the faster the fluid is sheared, the lower the apparent viscosity. Fluids that behave this way include paint, ketchup, blood, magma, mud, and other slurries and gels. The shear-thinning aspect of these fluids is important to the behavior of these fluids. For example, the viscosity of paint is reduced when it is sheared with a paint brush, and this allows the fluid to spread as a thin layer. The apparent viscosity increases when the shear stops, however, and this causes the viscosity to increase, which prevents the paint from running down a vertical surface to form messy drips. Alternatively, the shear-thinning attribute makes it possible for these fluids to flow through conduits with less head drop than expected. The high shear along the wall of the conduit reduces the apparent viscosity there, which increases the flow rate compared to what would occur for a Newtonian fluid. This characteristic is important for the intrusion of magma into thin dikes, or the flow of blood through arteries.

Fluids characterized by n>1 are called dilatant or shear-thickening fluids. These fluids stiffen or dilate when sheared. This behavior is less common than thixotropic, but a well known example is corn starch and water. This mixture will flow at slow strain rates, but it is rigid a high strain rates. There are many videos on-line (http://www.ksl.com/?nid=148&sid=2694253) showing people running across pools of corn starch—the shear thickening rheology allows people to walk on water. Another shear-thickening fluid is a mixture of silica and ethylene glycol (http://dilatantfluids.weebly.com/2-structure.html). This fluid is combined with kevlar to improve the performance of body armor

The Carreau model is also used to describe the rheology of non-Newtonian fluids, particularly polymers. The apparent viscosity of a Carreau fluid is given by

 (26)

## Boundary Conditions

Inlet: Conditions where fluid enters; Outlet: condition where fluid leaves the system; Wall: condition along a boundary

### Inlet Conditions

#### Velocity

The velocity normal to the boundary is specified as U0



The minus sign is there because a positive flux is outward away from the boundary by default.

#### Pressure, no viscous stress

The pressure is set to a specified value, and the shear stresses normal to the boundary are set to zero. This would occur where the boundary is connected to a large body of static fluid held at constant pressure

*P*=*P*o



#### Laminar Inflow

The fluid velocity distribution at the boundary is assumed to be distributed as if there were a conduit upstream of the boundary that is long enough for the flow to become fully developed. For example, this would create a parabolic boundary condition for a straight-walled conduit normal to the boundary.

### Wall

#### No Slip

This condition requires that the velocity at the wall is zero. This is the typical assumption for the velocity of fluid at walls.



#### Slip

This condition specifies that the flow normal to the boundary is zero, but it allows flow parallel to the boundary. This is also called a “no penetration” condition because the fluid cannot penetrate the boundary. This condition could occur for inviscid flows, where the effect of viscosity can be ignored. This also could be used where the effect of the boundary is to prevent fluid from leaving the model domain, but the boundary is far away from the region of interest.



#### Sliding or Moving Wall

This condition allows the wall to slide or move. Sliding is a special case where the movement is only in the tangential direction.

#### Leaking Wall

This boundary condition represents a porous wall where the fluid leaks out but otherwise the boundary behaves like a solid surface.